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ON THE OUTER AUTOMORPHISM GROUP OF A HYPERBOLIC GROUP

BY

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Dedicated to Professor Takushiro Ochiai for his sixtieth birthday

ABSTRACT

Let G be a one-ended, word-hyperbolic group. Let Γ be an irreducible lattice in a connected semi-simple Lie group of rank at least 2. If $h\colon \Gamma \to \mathrm{Out}(G)$ is a homomorphism, then $\mathrm{Im}(h)$ is finite.

1. Introduction

Let S be a compact, connected surface possibly with boundary. The mapping class group $\mathrm{Mod}(S)$ is the group of isotopy classes of homeomorphisms of S. When S is orientable, we consider only orientation-preserving maps. The following Farb–Kaimanovich–Masur rigidity theorem is shown for orientable surfaces in [11] and [6] (or see [2] for another proof). We show that the theorem holds for non-orientable surfaces as well by reducing the argument to the orientable case.

THEOREM 1: *Let N be a compact, connected surface, possibly with boundary. Let Γ be an irreducible lattice in a connected semi-simple Lie group (possibly with infinite center) of rank at least 2. Let $h\colon \Gamma \to \mathrm{Mod}(N)$ be a homomorphism. Then $\mathrm{Im}(h)$ is finite.*

Combining Theorem 1 and a deep result by Sela [16] regarding the structure of the outer automorphism group of a word-hyperbolic group, we immediately get the following result.

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COROLLARY 2: *Let G be a torsion-free, word-hyperbolic group. Suppose that G is freely indecomposable. Let Γ be an irreducible lattice in a connected semi-simple Lie group of rank at least 2. Let $h\colon \Gamma \to \mathrm{Out}(G)$ be a homomorphism. Then $\mathrm{Im}(h)$ is finite.*

To deal with word-hyperbolic groups with torsions, we have to understand the mapping class groups of 2-orbifolds. In Section 3 we show that Theorem 1 is valid for a hyperbolic 2-orbifold of finite volume as well (Theorem 8). It enables us to generalize Corollary 2 to the following form. By a theorem of Stallings, a group G which is not virtually cyclic is one-ended if and only if it does not split along a finite subgroup as an amalgamation or an HNN extension.

THEOREM 3: *Let G be a one-ended word-hyperbolic group. Let Γ be an irreducible lattice in a connected semi-simple Lie group of rank at least 2. Let $h\colon \Gamma \to \mathrm{Out}(G)$ be a homomorphism. Then $\mathrm{Im}(h)$ is finite.*

If G is two-ended, then it is virtually $\mathbb{Z}$, and the theorem is valid. The case that hyperbolic groups have more than two ends seems difficult. In particular we do not know if the theorem still holds when G is a free group of rank at least two.

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2. Proofs of Theorem 1 and Corollary 2

Theorem 1 is already proven when N is orientable in [11] and [6]. So we assume that N is non-orientable. We first treat the case when N is closed.

LEMMA 4: *Let N be a closed, non-orientable surface. Suppose N is not P^2 nor a Klein bottle. Let O be an orientable surface which is a double cover of N. Then there exists a finite index subgroup G in $\mathrm{Mod}(N)$ which is isomorphic to a subgroup in $\mathrm{Mod}(O)$.*

Proof: Let $\mathrm{Mod}^{\pm}(O)$ denote the group of isotopy classes of all (orientation preserving or reversing) homeomorphisms of O. Let i be an involution of O such that $O/i = N$. Let $S(O)$ be the subgroup in $\mathrm{Mod}^{\pm}(O)$ of the isotopy classes of homeomorphisms f of O with $f \circ i = i \circ f$. In [3], it is shown that there is an exact sequence such that

$$1 \to \mathbb{Z}/2\mathbb{Z} \to S(O) \xrightarrow{j} \mathrm{Mod}(N) \to 1.$$

$S(O)$ contains orientation reversing homeomorphisms of O. Take the subgroup $S'(O)$ in $S(O)$ of all orientation preserving maps. Then $S'(O)$ is a subgroup of $\mathrm{Mod}(O)$. It is known that $\mathrm{Mod}(O)$ contains a subgroup K of finite index which is torsion-free [10]. Put $G = S'(O) \cap K$. G has finite index in $S(O)$. Since G is torsion-free, from the exact sequence, we see that j is injective on G, so that G is a subgroup in $\mathrm{Mod}(N)$, which is of finite index. ∎

Having Lemma 4 we can prove Theorem 1 when N is closed.

PROPOSITION 5: *Let Γ be an irreducible lattice in a semi-simple Lie group of rank at least 2. Let N be a connected, non-orientable, compact surface. Let $h: \Gamma \to \mathrm{Mod}(N)$ be a homomorphism. If N is closed, then $\mathrm{Im}(h)$ is finite.*

Proof: If $N = P^2$ or Klein bottle, then the conclusion holds since $\mathrm{Mod}(N) = \{1\}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, respectively (see [3]). So suppose N is not P^2 nor Klein bottle. Let O be an orientable double cover of N. Then, by Lemma 4, there exists a subgroup G of finite index in $\mathrm{Mod}(N)$ which is isomorphic to a subgroup in $\mathrm{Mod}(O)$. Thus there exists a subgroup Γ' of finite index in Γ with $h(\Gamma') \subset G$, so that one obtains $h: \Gamma' \to \mathrm{Mod}(O)$. Since Γ' is also a lattice and O is orientable, by the Farb–Kaimanovich–Masur rigidity theorem, $h(\Gamma')$ is finite. This implies that $h(\Gamma)$ is finite. ∎

Now we discuss surfaces with boundary. Let S be a compact, connected surface, possibly with boundary. A map in $\mathrm{Mod}(S)$ generally permutes the components in the boundary of S. Let S° denote the interior of S, and $*$ a point in S°. Let $\mathrm{Mod}(S,*)$ denote the group of the isotopy classes (relative to $*$) of homeomorphisms of S which preserve $*$. Let $H(S)$ denote the group of homeomorphisms of S, and $H(S,*)$ the subgroup in $H(S)$ of homeomorphisms which preserve $*$. The following result is well-known (see for example [14]).

LEMMA 6: *Let M be a compact connected surface (possibly with boundary). Suppose M is not S^2, P^2, D^2, a Klein bottle, a torus, an annulus, a Moebius band. Then there exists the following exact sequence.*

$$1 \to \pi_1(M) \to \mathrm{Mod}(M,*) \to \mathrm{Mod}(M) \to 1.$$

If M is one of the surfaces excluded above, we have the following.

$$\pi_1(M) \to \mathrm{Mod}(M,*) \to \mathrm{Mod}(M) \to 1.$$

We start the proof of Theorem 1.

Proof: The theorem is known when N is orientable ([11], [6]). So assume that N is non-orientable. We work inductively on the number p of the boundary components of N. The conclusion holds when $p = 0$ by Proposition 5. Suppose that the theorem is true for p, and assume that the number of the boundary components of N is $p + 1$. Choose one boundary component of N and denote it B. Passing, if necessary, to a subgroup Γ' of finite index in Γ, we may assume that $h(\Gamma')$ leaves the component B invariant. Let M be the surface N with a 2-disk attached along B. M has p boundary components. Let $* \in M^\circ$. Then the subgroup of $\mathrm{Mod}(N)$ which leaves B invariant is isomorphic to $\mathrm{Mod}(M,*)$. So we may consider that $h(\Gamma') \subset \mathrm{Mod}(M,*)$.

First assume that M is not P^2, a Klein bottle, nor a Moebius band. By Lemma 6, there is the following exact sequence.

$$1 \to \pi_1(M) \to \mathrm{Mod}(M,*) \xrightarrow{i} \mathrm{Mod}(M) \to 1.$$

By the induction assumption, $i \circ h(\Gamma')$ is finite, so that, if necessary, passing to a subgroup Γ'' of finite index in Γ', $h(\Gamma'') \subset \mathrm{Ker}(i)$. Thus $h(\Gamma'') \subset \pi_1(M)$. Since Γ'' is a lattice and M is a surface, it follows that $h(\Gamma'')$ is finite. Hence $\mathrm{Im}(h)$ is finite.

Next assume that M is P^2, a Klein bottle, or a Moebius band. We have the following exact sequence.

$$\pi_1(M) \to \mathrm{Mod}(M,*) \to \mathrm{Mod}(M) \to 1.$$

Since both $\pi_1(M)$ and $\mathrm{Mod}(M)$ are virtually abelian in this case, and Γ' is a lattice, it is easy to see that $h(\Gamma') \subset \mathrm{Mod}(M,*)$ is finite. Hence $\mathrm{Im}(h)$ is finite. We have have proved Theorem 1. ∎

We prove Corollary 2.

Proof: Corollary 2 is proven in [7] under some extra assumption on "orientability" of G, and we follow the argument. Sela [16] showed, using his theory of JSJ decomposition, that $\mathrm{Out}(G)$ contains a subgroup A of finite index which is isomorphic to $\mathbb{Z}^n \times (\prod_i \mathrm{Mod}(S_i))$ such that $n \geq 0$ is an integer and S_i is a finite (possibly empty) collection of compact (orientable or non-orientable) surfaces, which the JSJ decomposition of G gives as "quadratically hanging surfaces".

Put $\Gamma' = h^{-1}(A)$. Γ' is a subgroup of finite index in Γ. It is a standard fact that on a lattice of the kind we discuss here, there is no non-trivial homomorphism to $\mathbb{Z}$ (by the Kazhdan–Margulis subgroup theorem). So $h(\Gamma')$ is contained in the direct factor $\prod_i \mathrm{Mod}(S_i)$ in A.

Let p_i be the projection from $\prod_i \mathrm{Mod}(S_i)$ to $\mathrm{Mod}(S_i)$. Apply Theorem 1 to each homomorphism $p_i \circ h$ and obtain that $p_i \circ h(\Gamma')$ is finite for all i, so that $h(\Gamma')$ is finite. So $\mathrm{Im}(h)$ is finite. ∎

3. Hyperbolic 2-orbifolds and proof of Theorem 3

Let G be a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^2)$. In general, $\Sigma = \mathbb{H}^2/G$ is a hyperbolic 2-orbifold and G is its orbifold fundamental group. $\pi\colon \mathbb{H}^2 \to \Sigma$ is a branched cover. If G is torsion free, then Σ is a 2-manifold. If G does not contain reflections, then the singular set of Σ is a finite collection of points.

In the following we always assume that Σ is a hyperbolic 2-orbifold of finite volume. Suppose Σ is orientable. G is a finitely generated fuchsian group. G does not contain reflections. Let E_i $(i = 1, \ldots, k)$ be the set of singular points on Σ with distinguished index e_i. Let $H(\Sigma)$ denote the group of orientation-preserving homeomorphisms $h\colon \Sigma \to \Sigma$ such that

$$h(E_i) = E_i \quad (i = 1, \ldots, k).$$

Let $\mathrm{Mod}(\Sigma)$ denote the group of isotopy classes of $H(\Sigma)$.

Let $\mathrm{Out}_+(G)$ be the orientation-preserving and type-preserving (see [9] for definition) outer automorphism group of G. By Theorem 1 in [9],

$$\mathrm{Mod}(\Sigma) \simeq \mathrm{Out}_+(G).$$

We have the same isomorphism when Σ is not orientable as well. In this case $\mathrm{Out}_+(G)$ is the type-preserving outer automorphism group of G. We can show it by following the argument given for the orientable ones in [9]. We only briefly outline it and point out where we have to do modification for the non-orientable case. If G contains reflections then it requires slightly different treatment in some steps in the argument. In this case Σ is a surface with totally geodesic boundary. Let E_0 be the set of singular sets on Σ for the reflections, and put $\Sigma_0 = \Sigma - \bigcup_{i=0}^k E_i$.

Pick a base point $x_0 \in \Sigma_0$. We can always isotope $f \in H(\Sigma)$ so that it fixes x_0. If $f \in H(\Sigma)$ is homotopic in Σ_0 to the identity, then f is isotopic in $H(\Sigma)$ to the identity (cf. Lemma 2 in [9]). This part relies on a theorem by Epstein ([5]), which is valid for orientable or non-orientable surfaces with or without boundary. This implies that $\mathrm{Mod}(\Sigma)$ is isomorphic to a subgroup of finite index in $\mathrm{Mod}(\Sigma_0)$ (Corollary 3 in [9]). Elements in $\mathrm{Mod}(\Sigma_0)$ may permute E_i's in general.

We define a homomorphism $\xi\colon \mathrm{Mod}(\Sigma) \to \mathrm{Out}_+(G)$. Let X be an arbitrary connected component of $\mathbb{H}^2 - \pi^{-1}(\bigcup_{i=0}^k E_i)$. If E_0 is empty there is only one component. $X \to \Sigma_0$ is a regular cover. Let $f \in H(\Sigma)$. Then $f: \Sigma_0 \to \Sigma_0$ lifts to a homeomorphism $f^*: X \to X$, which uniquely extends to $\mathbb{H}^2$. If E_0 is not empty, then we first extend f^* to the closure of X in $\mathbb{H}^2$, then to $\mathbb{H}^2$ using the reflections. Define φ by $\varphi(\gamma) = f^*\gamma(f^*)^{-1}$. Then $\varphi \in Aut_+(G)$. Since f^* is not unique, φ is determined up to inner automorphism. Now we define ξ: $\mathrm{Mod}(\Sigma) \to \mathrm{Out}_+(G)$ by $\xi([f]) = [\varphi]$, where $[f]$ is the isotopy class of f and $[\varphi]$ is the automorphism class of f. ξ is well-defined.

ξ is one-to-one. If f lifts f^* which induces an inner automorphism of Γ, then it will lift to a homeomorphism which induces the identity automorphism. Then, by a theorem of Marden ([13]), f is homotopic in Σ_0 to the identity. Marden stated his result only for the orientable case, but his argument also works for the non-orientable cases as well by a minor modification. One key ingredient is the following result (Lemma 1 in [13]).

LEMMA 7: *Suppose S is a Riemann surface and f is a homeomorphism of S. If there exists an arc γ from a point $* \in S$ to $g(*)$ such that α is homotopic to $\gamma g(\alpha)\gamma^{-1}$ for all simple closed curves α at $*$, then f is homotopic to the identity.*

For this result one can find an argument by Ahlfors in Section 6 in Bers [1], which works equally for our Σ. One remark is that even when E_0 is not empty, the assumption in Lemma 7 is required only for simple closed curves on $\Sigma - E_0$, which one can verify following Marden's argument. Finally, it follows from a theorem of Epstein [5] that f is in fact isotopic in Σ_0 to the identity.

ξ is onto. This is well known in the case that Σ is a manifold as a theorem of Fenchel-Nielsen. When Σ is an (orientable or non-orientable) 2-orbifold, one can find a proof by Macbeath (Theorem 3 in [12]). Macbeath has treated only the case when Σ is compact, but the same argument applies when Σ is of finite volume. The argument is based on the existence and the uniqueness of extremal quasi-conformal mappings for Riemann surfaces, which are known for not only compact ones but also ones of finite volume.

We obtained an isomorphism ξ: $\mathrm{Mod}(\Sigma) \to \mathrm{Out}_+(G)$ when Σ is non-orientable as well. In particular, $\mathrm{Out}_+(G)$ is isomorphic to a subgroup (of finite index) in $\mathrm{Mod}(\Sigma_0)$ when Σ is orientable or non-orientable. In general Σ_0 is not compact. We delete from Σ_0 a small open neighborhood at each point in E_i $(i = 1, \ldots, k)$ and an open cusp neighborhood at each cusp of Σ_0 such that those are mutually disjoint. We obtain a compact surface Σ_0' with $\mathrm{Mod}(\Sigma_0) = \mathrm{Mod}(\Sigma_0')$. We apply Theorem 1 to Σ_0' and obtain the following theorem.

THEOREM 8: *Let Σ be a hyperbolic 2-orbifold of finite volume. Let Γ be an* *irreducible lattice in a connected semi-simple Lie group of rank at least 2. Let* $h\colon \Gamma \to \mathrm{Mod}(\Sigma)$ *be a homomorphism. Then* $\mathrm{Im}(h)$ *is finite.*

We prove Theorem 3.

Proof: Let G be a one-ended, word-hyperbolic group. We use the JSJ decomposition due to Bowditch [4]. His theorem applies to all one-ended hyperbolic group. It gives a finite collection of "hanging" hyperbolic 2-orbifolds Σ_i of finite volume. Since G is one-ended, each puncture in Σ_i corresponds to an edge in the JSJ decomposition, so that an element in $\mathrm{Out}(G)$ gives only type-preserving elements for Σ_i. It follows that $\mathrm{Out}(G)$ has a subgroup of finite index which is a direct product of a free abelian group and the mapping class groups $\mathrm{Mod}(\Sigma_i)$. Therefore one can deduce Theorem 3 from Theorem 8 as before. ∎

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